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# Heavenly Mathematics

We're not ancient anymore. The birth and development of modern science have brought us to a point where we know much more about how the universe works. Not only do we know more; we also have reasons to believe what we know. We no longer take statements on faith. Experiments and logical arguments support us in our inferences and prevent us from straying into falsehood.

But how true is this, really? Do we really know, for instance, why the trajectory of a projectile is a parabola? In fact, anyone who has seen a soccer goalkeeper kick a ball downfield is aware that the ball's path is anything but symmetric. And yet, students accept their physics teachers' pronouncements about parabolas at face value—on authority. We trust our teachers to tell us the truth, just as we imagine medieval churchgoers accepting with blind faith the word of their priests. If we thought about it a little, we might recognize that air resistance is the culprit in the ball's divergence from a parabolic path. But do we know even this? Has anyone ever seen a soccer ball kicked in a vacuum?

It's impossible to live in our society (or any other) without taking some body of knowledge on authority. No one has the energy, or capacity, to check everything. We accept that the earth is a sphere (well, most of us anyway), without really knowing why. Only in one discipline—mathematics—is the "why" question asked at every stage, with the expectation of a clear and indisputable answer. Now, this is not the case in a lot of mathematics training in high school these days. Very few textbooks ask why $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. But this is the fault of modern textbooks and pedagogy, not of the subject itself. There is an explanation for this equation, and we'll see it in this chapter.

The goal of this chapter is twofold. Firstly, we will revisit topics in plane trigonometry in order to prepare for our passage to the sphere. But our second purpose takes precedence: to explore and learn without

taking anything on faith that we cannot ascertain with our own eyes and minds. This is how mathematics works, and by necessity it was how ancient scientists worked. They had no one to build on. Our mission is as follows:

*Accepting nothing but the evidence of our senses and simple measurements we can take ourselves, determine the distance to the Moon.*

Turning our eyes upward on a cloudless night, within a few hours we come to realize a couple of simple facts. The sky is a dome, perhaps the top half of a sphere, and we are at its center. (Don't forget, in this exercise we are not to accept the word of dissenting teachers and scientists!) The points of light on this hemisphere revolve in perfect circles around Polaris, the North Star, at a rate of one complete revolution per day (see plate 1).* By the disappearance of constellations below the horizon and their reappearance hours later, we may infer that the sky is an entire sphere (the *celestial sphere*), of which we can see only half at any one time.

But this observation does not narrow the possibilities regarding the shape of the Earth. Any planet that is sufficiently large with respect to its inhabitants will appear to be flat from their vantage point, discounting minor irregularities such as mountains and canyons. The most natural hypothesis is that the Earth is a flat surface (figure 1.1); it is also possible (although harder to imagine at first) that the Earth is a sphere or some other solid. How are we to choose?

Many of us have heard in school stories of those who believed in the flat Earth, perhaps even seen images from past sailors' nightmares: a ship sailing off an infinite waterfall at the edge of the Earth's disc. These often accompany tales of Christopher Columbus heroically attempting to convince the conservative Spanish court that the Earth is a sphere rather than a disc, making it possible to sail westward from Portugal to India. When I was a child, my teacher told me how a young Columbus, coincidentally about my age, discovered the curvature of the Earth. While watching a ship sail away from shore, Columbus noticed that its hull would be the first part to disappear, and eventually just before it vanished altogether, the only part left visible was the top of its mast (figure 1.2).

* You don't need to wait until nightfall. Several computer simulations of the night sky are available, including the free open source, multi-platform *Stellarium* (www.stellarium.org). The snapshots of the night sky in this book are generated using this program.
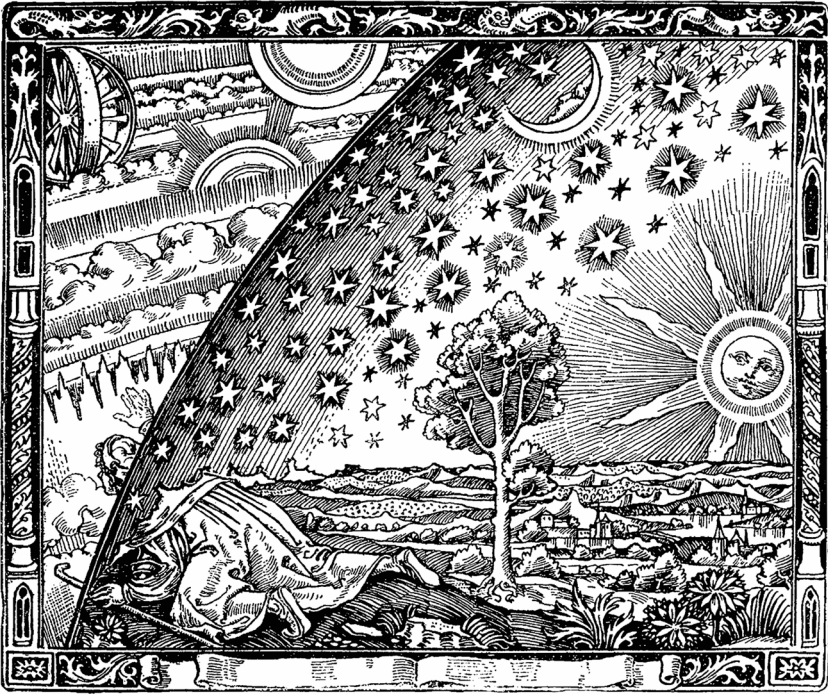
**Figure 1.1.** Engraving from Camille Flammarion's *L'Atmosphère Météorologie Populaire*, 1888. Source: http://en.wikipedia.org/wiki/File:Flammarion.jpg.
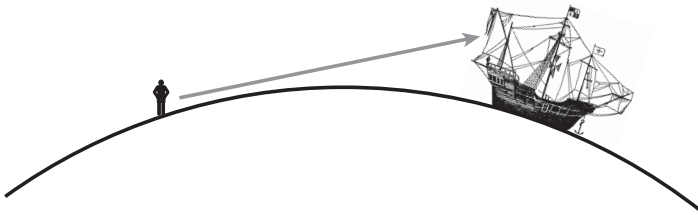
**Figure 1.2.** Columbus's line of sight as a ship sails away.

All of this is fiction. Columbus was trying to convince the Spanish court of the Earth's *size*, not its *shape*. In fact, Columbus thought the Earth was smaller than it actually is, and he fortuitously came upon the West Indies approximately where he thought the East Indies were supposed to be. His error was caused, in part, by his use of an Arabic estimate for the length of a degree of latitude, which he assumed was in

Roman miles, but in fact was in Arabic miles. Moral of this story: watch your units. The story of the ship disappearing below the horizon that my teacher attributed to Columbus is actually 1500 years older, in the Greek scientist Strabo's writings around the time of the birth of Christ. Centuries before that, Aristotle had given several arguments for the sphericity of the Earth, including the observation that the shadow cast by the Earth on the Moon during a lunar eclipse is always a circle.

I should have known that my teacher was telling a story. Who else but sailors would be the first to notice how ships disappear below the horizon? Ever since Aristotle, hardly any observant people, whether navigators, theologians, or scholars, have considered the Earth to be flat. The modern myth of ancient belief in the flat Earth was popularized by the 19th-century novelist Washington Irving in an imaginative biography of Columbus (figure 1.3). Historians of science have been trying (mostly

**Figure 1.3.** Columbus arrives at the New World, in Washington Irving's *The Lives and Voyages of Christopher Columbus*, Chicago: Donohue, Henneberry & Co.

unsuccessfully) to curb its spread ever since. So we shall accept what we now know are ancient ideas that the Earth is round, and turn our exploration of the universe away from shape and toward size.

## How Large Is the Earth?

Obviously we cannot determine the dimensions of the Earth by measuring it directly, but there are several indirect approaches. The most renowned historical method, by 3rd-century BC astronomer and mathematician Eratosthenes of Cyrene, involves observing rays of sunlight penetrating well shafts in different locations. We shall follow instead a scheme devised by the great scholar Abū al-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī (AD 973–1050?; figure 1.4). One of the most prolific authors of the medieval period, al-Bīrūnī wrote at least 146 treatises on almost every area of science known in his time, including mechanics, medicine, and mineralogy in addition to mathematics and astronomy. One of his most famous works describes social and religious practices, geography, and philosophy in India. His *Kitāb Taḥdīd al-Amākin* (or *Book on the Determination of the Coordinates of Cities*) was inspired originally by the problem of finding the *qibla*—the direction of Mecca, toward which Muslims must face to pray. Since it's just as easy to find the direction to some location other than Mecca, the book is actually a comprehensive description of mathematical techniques of locating cities on the Earth's surface. Since our goal, here
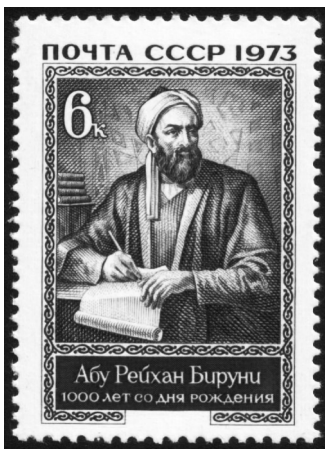
**Figure 1.4.** A portrait of al-Bīrūnī on a Soviet postage stamp.
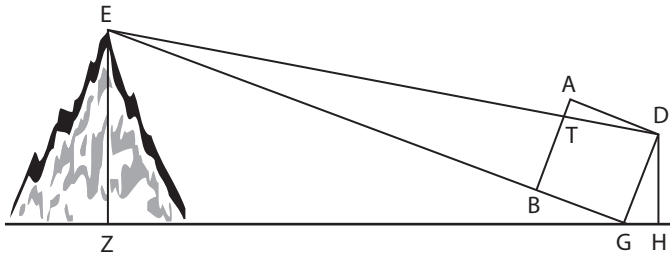
**Figure 1.5.** Al-Bīrūnī's determination of the height of a mountain.

and elsewhere, is not primarily to represent the historical text faithfully but rather to clarify the argument, we will simplify the mathematics and use modern functions and notation.

Bīrūnī begins by determining the height of a nearby mountain (near Nandana, in northern Pakistan). This isn't as easy as it sounds, since the point at the mountain's base is buried under tons of rock (figure 1.5). He builds a square *ABGD*; since he does not tell us how big it was, we set the square's side length equal to 1 meter for the sake of convenience. He then lines up the square so that the sight line along its bottom, *GB*, touches the top of the mountain *E*. Let *H* be the perpendicular projection of *D* onto the ground, and let *T* be the intersection of *AB* with *DE*. Using our meter stick we measure $GH = 5.028$ cm and $AT = 0.01648$ cm. Clearly it's impossible to measure such a short distance with such accuracy; the fact that Bīrūnī was able to get a reasonable value for the Earth's size suggests that his square must have been huge.

Now we use similar triangles. From $GE/GD = AD/AT$ we compute $GE = 6067.96$ meters, and from $EZ/GE = GH/DG$ we find the mountain's height to be $EZ = 305.1$ meters. Not exactly a colossus, which is just as well, since our next task is to climb it.

Once we have reached the top of the mountain, we look to the horizon. With good enough instruments we should notice that the horizon is not precisely horizontal to us, but dips slightly downward (figure 1.6). Bīrūnī tells us that he measured the value $\theta = 34' = 34/60^\circ = 0.56667^\circ$ for this angle, which is very small, but likely just within his capacity to measure. We know that $\theta$ is also $\measuredangle TOZ$ at the center of the Earth, and that the radius is $r = OT = OZ$. Now, since $\Delta OTE$ is a right triangle, we have

$$\cos\theta = \frac{OT}{OE} = \frac{r}{r + 305.1\text{ m}}.$$
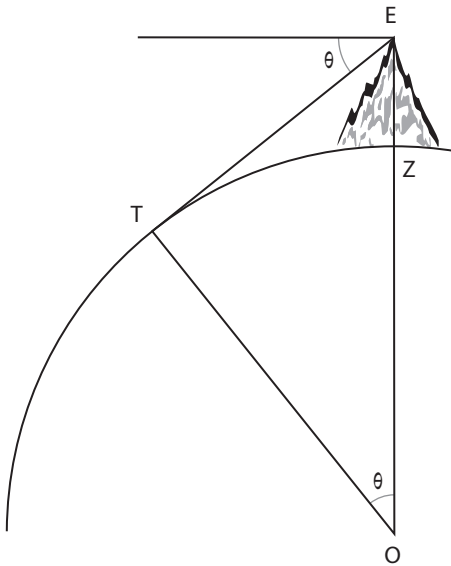
**Figure 1.6.** Al-Bīrūnī's determination of the radius of the Earth.

But we know $\theta$, so the left side of the equation is $\cos 0.56667^\circ$, and the only unknown on the right side is $r$. Solving for $r$, we find that the Earth's radius is 6238 km. (There is a delicate matter hidden in this solution, however: a minute change in the value for $\theta$ results in a large change in the value of $r$. One wonders how al-Bīrūnī pulled off the accuracy that he did.) Multiplying by $2\pi$, we get a value for the Earth's circumference of 39,194 km. Its actual value is about 40,000 km. Not bad (in fact, maybe a little *too* good) for a process with its share of crude measurements!

## Building a Sine Table with Our Bare Hands

There's a problem in the last step of our procedure. Our goal was to work without relying on anyone or anything, and at the end we likely relied on Texas Instruments to tell us the value of $\cos 0.56667^\circ$. This violates our rules, so to do this properly we must find a way to compute trigonometric values without technological assistance. Again we will follow the ancient and medieval astronomers (adopting a few modern simplifications). Our mission is to compute a table of sines, since every other trigonometric function can be calculated once we have a sine table. So, we must find the sine of every whole-numbered angle between 1° and 90°.

If an angle that we come across in our astronomical explorations isn't a whole number, we'll just trust that we can interpolate within our table.

The first person whose trigonometric table comes down to us today was the 2nd-century AD Alexandrian scientist Claudius Ptolemy. His astronomical masterpiece, the *Mathematical Collection*, contains a remarkable collection of models for the motions of the heavenly bodies. It is known today mostly for being wrong—it places the Earth at the center of the universe. But it was one of the most successful scientific theories of all time, dominating astronomy for a millennium and a half under its Arabic title *Kitāb al-majistī* ("The Great Book"), the *Almagest*.

The first of the *Almagest*'s 13 books contains a description of how one can build a trigonometric table with one's bare hands. (Ptolemy actually used another function called the chord, but the chord is so similar to the sine that we won't distort much by sticking with the sine.) Several sine values, the ones we remember from memorizing the unit circle in high school, may be found immediately. Figure 1.7 shows how to find $\sin 30^\circ$ and $\sin 45^\circ$. For $\sin 30^\circ$ we notice that the triangle obtained by reflecting the original triangle about the horizontal axis is equilateral, which makes $\sin 30^\circ = 0.5$. For $\sin 45^\circ$, note that the horizontal and vertical sides of the key triangle are equal; applying Pythagoras gives us the result $\sin 45^\circ = \sqrt{1/2} = 0.7071$.

We now have two of the 90 values we need for our sine table; if we count $\sin 90^\circ = 1$, we have three. There is a long way to go. But the Pythagorean Theorem tells us that
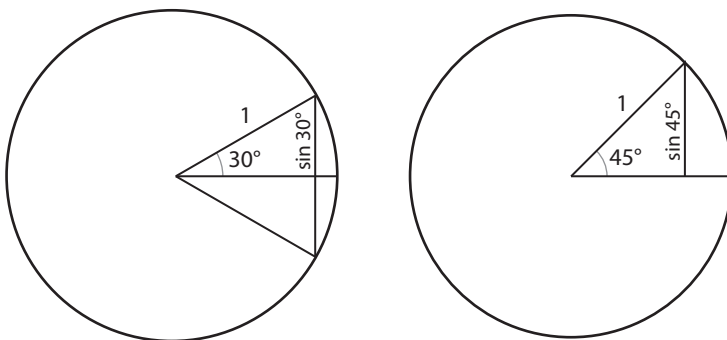
**Figure 1.7.** The sines of 30° and 45°.

$$\sin^2\theta + \sin^2(90° - \theta) = 1,$$

so we can always find $\sin(90° - \theta)$ if we know $\sin\theta$. This fact effectively cuts our task in half . . . but half of a huge task is still daunting.

**For readers in a hurry, this arrow means that the mathematics contained here may be bypassed without losing the thread of the story.**

→Our next value, $\sin 36°$, does not come from the memorized unit circle. Ptolemy finds it using Euclid's construction of a regular pentagon; we will use the same shape, but a slightly different path. Consider the "star" configuration in figure 1.8. Let's assume that the sides of the regular pentagon have length 1. Since the shape inscribed in the circle is a regular pentagon, $\measuredangle B$ in $\Delta ABC$ is 108°. (To see this, note that a pentagon can be partitioned into three triangles, so the sum of the five equal pentagon angles is $3 \times 180° = 540°$.) But by symmetry the other two angles in this triangle are equal to each other, so $\alpha = \beta = 36°$. This means that our goal, $\sin 36°$, is $BF$. By symmetry, $\measuredangle ABD = 36°$, which leaves $\gamma = 108°$, $\delta = 72°$, and
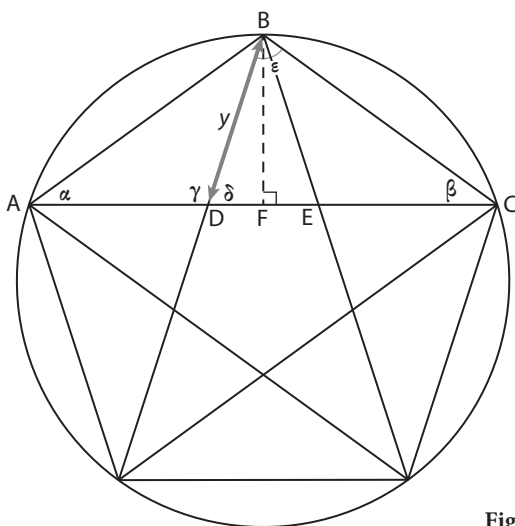
**Figure 1.8.** The derivation of sin 36°.

$\varepsilon = 72^\circ$. So $\Delta BCD$ is isosceles, and $\Delta ABD$ is similar to $\Delta ABC$. This allows us to determine length $y = BD$, since

$$\frac{BD}{AB}\left(=\frac{y}{1}\right) = \frac{AB}{AC} = \frac{1}{AD+1} = \frac{1}{y+1}.$$

By cross multiplication $y^2 + y = 1$, and this quadratic equation surprisingly produces $y = 0.61803$, the golden ratio! From here it's downhill to $\sin 36^\circ$. We know that $DF = \frac{1}{2}AC - AD = \frac{1}{2}(1+y) - y = 0.19098$, and so from Pythagoras, $BF = \sin 36^\circ = 0.58779$.→

We now have the sines of 30°, 36°, 45°, 54° (by Pythagoras), 60°, and 90°. It's time to accelerate things a bit. Can we come up with a systematic tool that finds more than one sine value at a time? The sine addition law is just the ticket, and Ptolemy demonstrates an equivalent to it next.

**Theorem:** If $\alpha, \beta < 90^\circ$, then $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

The condition in this theorem isn't really necessary, but we won't bother generalizing. (Another way of saying this is that we leave that task to the reader.) And of course, in the process of discovery we never know the result in advance. So we'll proceed as if the above had never been written and simply seek a formula for $\sin(\alpha+\beta)$, following the proof that was included in most trigonometry textbooks in the first half of the 20th century.

→**Proof:** In figure 1.9, since $OD = 1$, the quantity we're after is $GD = \sin(\alpha+\beta)$. It is conveniently broken into two parts, $GF$ and $FD$. Now from $\Delta OCD$ we know that $OC = \cos\beta$ and $CD = \sin\beta$. So, in $\Delta OCE$ we now know the hypotenuse. Thus $\sin\alpha = EC/\cos\beta$, so $EC = \sin\alpha\cos\beta$. Since $EC = FG$, we're halfway there: we've found one of the two line segments comprising $GD$.

We can find $FD$ by noticing first that $\Delta OCE$ is similar to $\Delta DCF$. This statement is true because $\measuredangle FCO = \alpha$, so $\measuredangle FCD = 90^\circ - \alpha$, and the two triangles share two angles, so they must share the third. So $\measuredangle FDC = \alpha$ . . . and we already know the hypotenuse $CD$ of $\Delta DCF$. So $\cos\alpha = FD/\sin\beta$, which gives $FD = \cos\alpha\sin\beta$, and finally we have $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. QED→
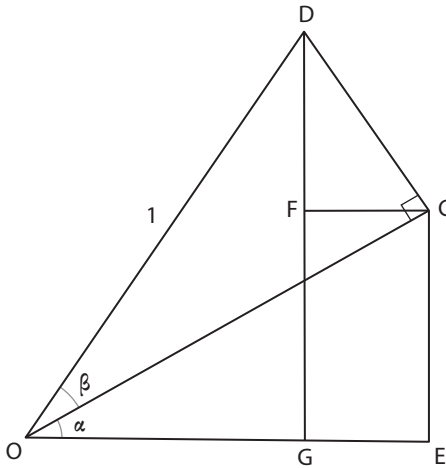
**Figure 1.9.** The proof of the sine addition law.

Perhaps for the first time in our mathematics education, we have a reason to believe the sine addition law. That is valuable in itself, but more important is the use to which we will put it. Just as Ptolemy did, we may use this theorem to calculate the sine of the sum of two angles for which we already know the sines. For instance, from $\sin 30°$ and $\sin 45°$ we can calculate $\sin 75°$, or by substituting $\sin 36°$ for both $\alpha$ and $\beta$, we have $\sin 72°$.

A similar process (explored in the exercises) allows us to derive the formula for the sine of the difference between two angles,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

So we can, for instance, use our values for $\sin 72°$ and $\sin 75°$ to find $\sin 3°$. And from this step, using the sine addition law repeatedly, we can find the sines of *all* multiples of 3°. But now Ptolemy reaches an impasse. Even with an extra theorem—the sine half-angle identity ($\sin\alpha/2 = \sqrt{(1-\cos\alpha)/2}$, explored in the exercises)—he is unable to find the sine of any whole-numbered angle that is *not* a multiple of 3°!

The problem of passing from $\sin 3°$ to $\sin 1°$, an example of the famous Greek conundrum of trisecting the angle with ruler and compass, troubled many astronomers after Ptolemy. In fact, getting an accurate value for $\sin 1°$ was more important than finding a value for $\pi$. After all, while $\pi$ comes up every once in a while when predicting the movements of the
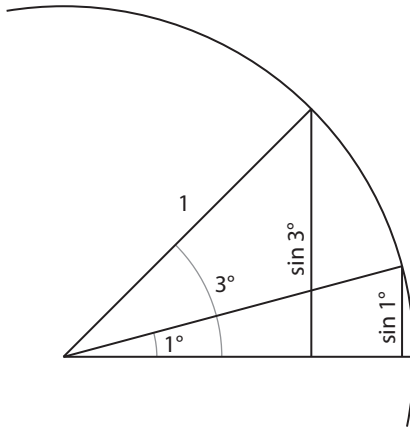
**Figure 1.10.** sin 3° < 3 sin 1°.

stars and planets, sine values appear all the time. So the entire edifice of predictive astronomy relied mathematically on this one, geometrically unattainable, value.

Since Ptolemy was unable to use geometry, he turned to approximation. If you consider the sines of 1° and 3° (drawn, not to scale, in figure 1.10), it's clear that $\sin 3^\circ$ is greater than $\sin 1^\circ$, but it's not three times as big. Due to the gradual leveling off of the circle as one works upward from its rightmost point, the sine increases at a slower and slower rate as the angle increases. Said more generally:

**Theorem:** If $\beta < \alpha < 90^\circ$, then $\dfrac{\alpha}{\beta} > \dfrac{\sin\alpha}{\sin\beta}$.

Now, using the half-angle formula we can follow in Ptolemy's footsteps and calculate from $\sin 3^\circ$ the values of $\sin\frac{3}{2}^\circ$ and $\sin\frac{3}{4}^\circ$. These numbers are the key, for now we can apply our new theorem to get bounds on $\sin 1^\circ$: first, substitute $\alpha = 1^\circ$ and $\beta = \frac{3}{4}^\circ$; this produces $\frac{1^\circ}{\frac{3}{4}^\circ} > \frac{\sin 1^\circ}{\sin\frac{3}{4}^\circ}$, which simplifies to $\sin 1^\circ < \frac{4}{3} \cdot \sin\frac{3}{4}^\circ = 0.01745279$. Next, substitute $\alpha = \frac{3}{2}^\circ$ and $\beta = 1^\circ$; this gives the lower bound $\sin 1^\circ > \frac{2}{3} \cdot \sin\frac{3}{2}^\circ = 0.01745130$. Combine the results, and we get

$$0.01745130 < \sin 1^\circ < 0.01745279.$$

If we hope for our table to be accurate to five decimal places, then we have our sought-after value: 0.01745. (If we need more precision, then we have a problem, although medieval astronomers did find ways of

extending Ptolemy's method to generate more accuracy.) From this point we can fill in the rest of our table, just by applying the sine addition and subtraction laws to $\sin 1^\circ$ and the sines of the multiples of 3°.

Ptolemy does not tell us what he thought of being forced into the sordid world of approximation to find $\sin 1^\circ$. But we do know that at least two later scientists objected strenuously to bringing numerical methods into the pure, unsullied world of geometry. The 12th-century Iranian Ibn Yaḥya al-Maghribī al-Samawʾal was so aggrieved by it that he included Ptolemy in his *Exposure of the Errors of the Astronomers*, and actually constructed his own sine table with 480° in a circle rather than 360°. Giordano Bruno, the 16th-century theologian and philosopher who was eventually burned at the stake (although not for this reason), felt that the entire discipline of trigonometry was undermined and proclaimed, "Away with the useless tables of sines!"

As odious as approximation was to these two scientists, the methods we have just seen were the mathematical basis of all trigonometric tables through the 16th century. The most prodigious set of trigonometric tables in early Europe, the *Opus palatinum*, was composed by Georg Rheticus, who had been the leading early champion of Nicolas Copernicus's Sun-centered universe. Rheticus died in 1574 before his work was completed, but the tables were completed and published in 1596 by Lucius Valentin Otho. The 700 large pages comprising the second half of Rheticus and Otho's massive volume contain tables of all six trigonometric functions to ten decimal places for every 10″ of arc (figure 1.11). In modified form, they were the dominant trigonometric tables used by scientists until they were replaced, finally, in 1915. But the methods Rheticus used to generate these tables were at heart no different from those of Claudius Ptolemy, one and a half millennia before.

This is not to say that better methods had not been considered. Only 150 years before Rheticus but in a different culture, the Persian astronomer Jamshīd al-Kāshī had considered the sin 1° problem in a very different way. Al-Kāshī was a natural for this attack: he was a master calculator, and his fame rests partly on computing $\pi$ to the equivalent of 14 decimal places—twice as many as any of his predecessors. He didn't stop there. His first attempt on sin 1° was an extension of Ptolemy's method, but later he took an entirely different tack. It begins with a consideration of the sine triple-angle formula,
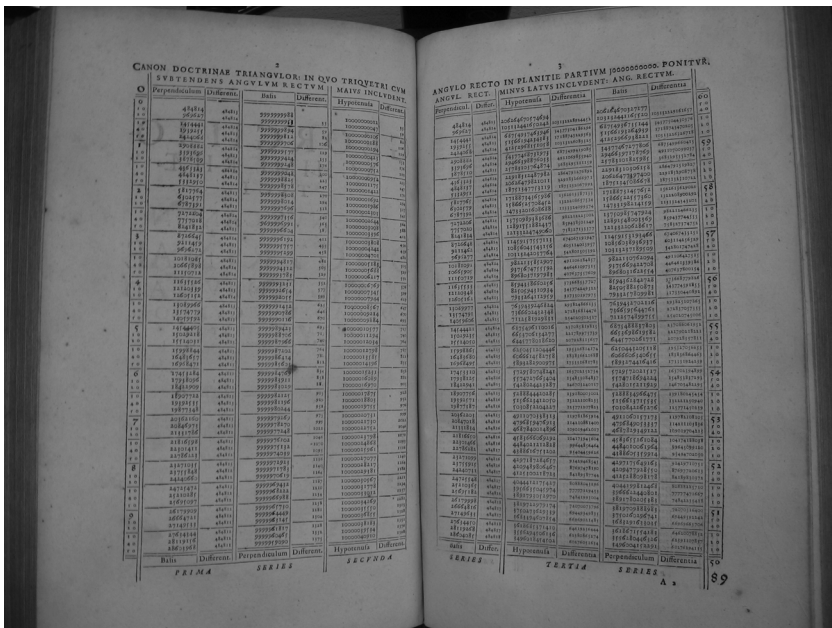
Figure 1.11. The first of 700 pages of Rheticus and Otho's trigonometric tables in the *Opus palatinum*. Each pair of columns represents one of the six major trigonometric functions. Courtesy of the Burndy Library, MIT.

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

which we leave to the interested reader to verify (use the sine addition law on $\sin(\theta+\theta+\theta)$). Substitute $\theta = 1^\circ$, and we have a cubic equation whose solution is the sought-after $\sin 1^\circ$:

$$\sin 3^\circ = 3\sin 1^\circ - 4\sin^3 1^\circ.$$

But the cubic would not be solved for another 125 years and far from Persia, by Gerolamo Cardano in 1545. Clearly, al-Kāshī could not wait that long.

Instead, he found a way to determine the solution one digit at a time, not descending brazenly into approximation but bypassing geometry altogether, using a method something like the following.

→Let $x = \sin 1^\circ$; then $\sin 3^\circ = 3x - 4x^3$. With a little rearrangement, we arrive at $x = \frac{\sin 3^\circ + 4x^3}{3}$. Now visually, what we're looking for is the place where the graphs $y = x$ and $y = \frac{\sin 3^\circ + 4x^3}{3}$ cross each other (figure 1.12). Take an initial guess at the solution; an obvious
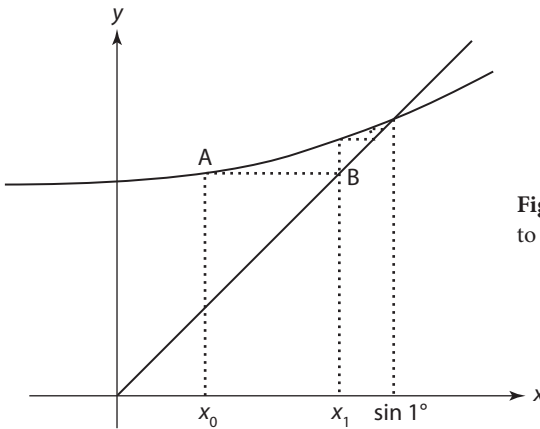
**Figure 1.12.** Fixed point iteration to find sin 1°.

choice is $x_0 = \frac{1}{3}\sin 3° = 0.017445319$. Plug it into the right side of our equation and we get 0.017452397, corresponding to the vertical distance from $x_0$ to $A$ on the graph. We treat this new value as our next guess $x_1$. On the graph, this means that we must convert the height $x_0A$ into an $x$-coordinate. We can take this step by moving horizontally from $A$ to $B$, where we know that $y = x$; then we move down to $x_1$.→

From here we simply repeat the process as many times as desired. Plugging $x_1$ into the right side of the equation yields $x_2 = 0.017452406$; another iteration yields an identical value for $x_3$, to nine decimal places. So already we have nine decimal places for $\sin 1°$, with an easy method at hand to generate as much accuracy as any numerical stickler may demand. Al-Kāshī stopped at the equivalent of 16 decimal places. This technique, today called *fixed point iteration*, is not guaranteed to work with every equation of this sort, but fortunately it works extremely efficiently in our case. And from our value of $\sin 1°$, we may now fill in the rest of the sine table, with as much precision as we have patience.

## The Distance to the Moon

The computational energy required to construct a sine table using the above methods is hardly a trivial matter; we caution the reader not to

try this at home without a lot of free time. Now that we know *how* to do it, we shall assume that the reader has put in the required years of drudgery, and lying before us is a complete set of trigonometric tables, ready to be used for our astronomy. We have taken a long diversion to determine the single cosine value that al-Bīrūnī needed to complete his determination of the circumference of the Earth, but the good news is that the diversion is needed only once. We may now press on, assured that whenever we need a trigonometric value, we may simply look it up.

It is one thing to calculate the size of the Earth, but another task entirely to venture beyond the Earth's surface to find the distance to the Moon. In fact this feat has been accomplished frequently; Ptolemy himself came to an accurate value already in the 2nd century AD. We mention only in passing that he also calculated the distance to the Sun, and came up with a value about 19 times too small. His method was sound, even if his result was not.

The key is *parallax*: the fact that two observers, in different places, will see the same object in different positions with respect to a distant background. In the case of the Moon the distances are vast, but the principle still applies. Figure 1.13 shows the Moon in the night sky at the same moment from two different locations; the change in its position within the constellation Scorpius is clear. This is the sort of observation that Ptolemy used. (In his calculation of the Sun's distance, the error was his assumption that the Sun's parallax was just on the edge of being
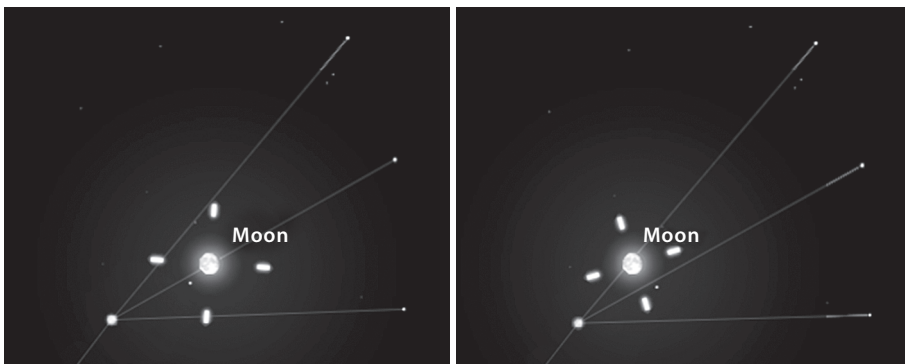
**Figure 1.13a and 1.13b.** The Moon as seen from Vancouver, Canada in (a) and from London, England in (b) on April 30, 2010. In Vancouver the Moon is on the middle of the three claws of Scorpius, in London it is on the upper claw.
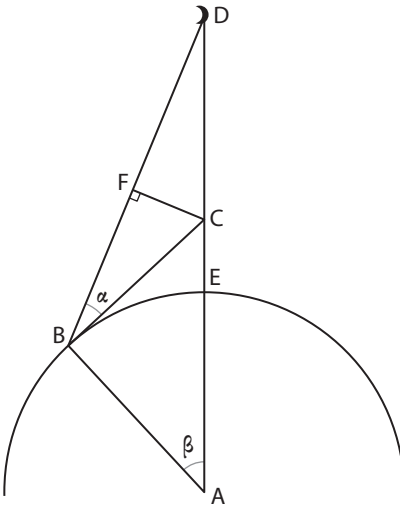
**Figure 1.14.** Finding the distance to the Moon.

observable with the naked eye. Actually, the parallax is much smaller than that.)

Although our method is simpler than Ptolemy's, the idea is the same. We assume that for one observer, $E$ in figure 1.14, the Moon is directly overhead; so, its altitude is 90°. For a second observer, 300 km away at $B$, the Moon's altitude is $\alpha = 87.201°$. Now, without telephones it would be difficult to make sure that the two observations take place at the same moment. One way around this is to observe during a lunar eclipse, which takes place simultaneously for all Earthly observers.

→These are all the observations we need. Since the value we found earlier for the Earth's radius is 6238 km, we know that angle $\beta$ is $300/(2\pi \cdot 6238)$ of a circle, or 2.7555°. Next we work our way up the figure. Using $\Delta ABC$ we find that $BC = AB \tan\beta = 300.23$ km, and that $r/(r + CE) = \cos\beta$, from which we find $CE = 7.2209$ km. Next, using $\Delta BCF$, we calculate $CF = BC \sin\alpha = 299.87$ km. The most important observation follows: since the three angles at $C$ add up to 180°, we know that $\angle DCF = \alpha + \beta = 89.957°$. Now we can use $\Delta CDF$ to find $CD = CF/\cos 89.957° = 395{,}160$ km. Add to this the inconsequential 7 km that is $CE$, and our value for the Moon's distance is 395,167 km. (The correct distance is around 384,400 km).→

Just for fun, let's see what we can determine from this result about the dimensions of the solar system. If we were to shrink the universe so that the Earth is the size of a soccer ball, its radius would be about 11 cm. Since we know that the Moon's distance is 395,167 km and the Earth's radius is 6238 km, the Moon's distance in our soccer ball universe is $11 \cdot 395{,}167/6238 = 695$ cm, or about 7 meters—about half the distance across a typical classroom. The Moon would be about the size of a tennis ball, with a radius of 3 cm. Let's step for a moment beyond what the ancients were capable of observing. In this scale, the Sun's diameter would be about 24 m, about the height of an eight-story building, and would be about 2.6 km away. The nearest star, Proxima Centauri, would have a diameter of only 3.5 m, about one story high. It would be about 700,000 km away, almost twice the actual distance from the Earth to the Moon. Our galaxy consists almost entirely of empty space.

We have completed our mission to find the distance to the Moon using only simple measurements. At the same time we've refreshed our plane trigonometry and become accustomed to the "prove it to me" attitude that mathematics requires. With these experiences under our belts, it is time to turn to the sphere.

## Exercises

1. Using only a basic pocket calculator (no scientific calculators, although you may take square roots), determine the value of $\sin 3^\circ$ in the most efficient way that you can. Include in your work the computation of any sine values you need along the way.
2. The sine subtraction law is $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$.
   (a) Derive this result by replacing $\beta$ with $-\beta$ in the addition law.
   (b) Now attempt the more interesting task: prove it geometrically using figure E-1.2.

**Figure E-1.2.**



3. (a) Show by construction that $2\sin A > \sin 2A$.
   (b) Given two angles $A$ and $B$ ($A + B$ being less than 90°), show that $\sin(A + B) < \sin A + \sin B$.
   [Wentworth 1894, p. 8]